3D wave field reconstruction from intensity-only data: variational inverse imaging techniques

(Invited Paper)

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***Abstract*—We present a variational approach to obtain a reconstruction of module and phase of a 3D wave field from intensity-only measurements on two or more sensor planes at different axial positions. The objective functional consists of a data fidelity term and a regularizer. The fidelity term corresponds to the likelihood function derived for the Gaussian noisy observations of the wave field intensities (powers). The wave field reconstruction is framed as a constrained nonlinear optimization with respect to a 2D object wave field and is based on the augmented Lagrangian technique. The main goal is to design an algorithm which is more efficient and accurate than the conventional ones such as the well-known Gerchberg-Saxton algorithms and their multiple modifications. As a further development we discuss a variational approach using a transform domain prior on phase and module of the 2D object wave field.**

I. Introduction

A new recursive augmented Lagrangian (AL) algorithm is presented for reconstruction of a module and a phase of 3D wave field from intensity-only measurements on two or more sensor planes at different axial positions. The wave field reconstruction is formulated as a constrained nonlinear optimization allowing to involve a prior information (simpler the prior) on wave field of interest. The main intention is to design the algorithm which is more accurate than the conventional ones such as the well-known Gerchberg–Saxton algorithms and their multiple modifications (e.g. [1]-[4]).

The considered problem is specified as follows. Let $u_0(x)$ and $u_r(x)$, $r = 1, ..., l$, denote complex-valued wave field distributions in the object and observation (sensor) planes, respectively, given in lateral coordinates $x \in R^2$. The index r corresponds to a distance z_r between the parallel object and rth observation planes, and l is a total number of the observation planes.

In discrete modeling all continuous variables are pixelated with the argument x replaced by the digital one with the following replacements of the continuous distributions by their discrete counterparts: $u_0(x) \to u_0[k]$, $u_r(x) \to u_r[k]$.

The discrete intensity observations are given in the form

$$o_r[k] = |u_r[k]|^2 + \varepsilon_r[k],\ r = 1, ..., l, \tag{1}$$

where the wave field intensity (power) is measured with an additive random errors $\varepsilon_r[k]$. We assume that this random noise is zero-mean Gaussian, $\varepsilon_r[k] \sim \mathcal{N}(0, \sigma_r^2)$.

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The problem is to reconstruct pixelated complex-valued wave field distributions $u_o[k]$ and $u_r[k]$ at the object and sensor planes from the noisy data (1). This distribution (image) restoration is known as an inverse problem. It is convenient to denote the complex-valued wave fields as vectors in R^n by concatenating their column and use bold lower case characters for these vectors. Then, the wave field propagation from a diffraction (object) plane with a complex-valued distribution $\mathbf{u}_0$ gives a complex-valued distribution $\mathbf{u}_r$ in the rth image (sensor) plane as

$$\mathbf{u}_r = \mathbf{A}_r \mathbf{u}_0, \tag{2}$$

where $\mathbf{A}_r$ is a forward propagation operator from the object to rth plane represented in the matrix form.

We consider a coherent light scenario with a paraxial wave field propagation model based on the Rayleigh-Sommerfield equations. The operator $\mathbf{A}_r$ in (2) is specified by discretization of this modeling. It can be convolutional single or double size model, angular spectrum decomposition (ASD) [5], or recent discrete diffraction transforms in matrix (M-DDT) [6] or frequency (F-DDT) forms [7]. These DDT models are obtained for the Fresnel approximation of the Rayleigh-Sommerfield integral and enable an accurate pixel-to-pixel mapping of the pixelated $\mathbf{u}_0$ to $\mathbf{u}_r$.

With the vector-matrix notation (2) the observation equation (1) takes the form

$$\mathbf{o}_r = |\mathbf{u}_r|^2 + \boldsymbol{\varepsilon}_r,\ r = 1, ..., l, \tag{3}$$

where the modulus $|\cdot|$ and square $|\cdot|^2$ are the point-wise operations applied to the elements of the corresponding vectors.

II. Variational wave field reconstruction

In this article, we do not follow any variant of Gerchberg-Saxton or Fienup's error-reduction algorithms [1], [2], [8] but rather apply the maximum likelihood style approach. For the Gaussian noise distribution and the observation model (3) it results in the following criterion

$$J = \sum_{r=1}^{l} \frac{1}{2\sigma_r^2} ||\mathbf{o}_r - |\mathbf{u}_r|^2||_2^2 + \mu \cdot pen(\mathbf{u}_0), \tag{4}$$

where the norm $||\cdot||_2^2$ is Euclidean and the power in $|\mathbf{u}_r|^2$ is an element-wise operation. The first summand in (4) is obtained as the main term of the minus logarithm of the Gaussian likelihood function corresponding to the observation model (3), and the second summand is the penalty (regularization) including the prior on the object distribution $\mathbf{u}_0$ to be reconstructed.

The wave field reconstruction is formulated as the following constrained optimization problem

$$\hat{\mathbf{u}}_0 = \arg\min_{\mathbf{u}_0} \sum_{r=1}^{l} \frac{1}{2\sigma_r^2} ||\mathbf{o}_r - |\mathbf{u}_r|^2||_2^2 + \mu \cdot pen(\mathbf{u}_0) \quad (5)$$

subject to $\mathbf{u}_r = \mathbf{A}_r \mathbf{u}_0$.

with the only unknown variable $\mathbf{u}_0$ and the distributions $\mathbf{u}_r$ calculated according to the forward propagation models (2).

The parameter μ in (5) defines a balance between the accuracy of the observation fitting and the prior. If $\mu = 0$ the solution $\hat{\mathbf{u}}_0$ minimizes $\sum_{r=1}^{l} \frac{1}{2\sigma_r^2} ||\mathbf{o}_r - |\mathbf{u}_r|^2||_2^2$ ignoring the fact that the data $\mathbf{o}_r$ are noisy. It can result in noisy and non-smooth $\hat{\mathbf{u}}_0$. If $\mu > 0$ and large then the noise effects are well suppressed but the solution $\hat{\mathbf{u}}_0$ can be oversmoothed with important features lost. A proper selection of μ known as a regularization parameter is an important point of the variation formulation in inverse imaging.

III. Augmented Lagrangian (AL) Algorithm

The Augmented Lagrangian Method, introduced independently by Hestenes [9] and Powell [10] is now classical for the minimization of functionals in presence of linear equality constraints.

The Augmented Lagrangian corresponding to (5) is of the form

$$L(\mathbf{u}_0, \{\mathbf{u}_r\}, \{\mathbf{\Lambda}_r\}) = \quad (6)$$

$$\sum_{r=1}^{l} \frac{1}{\sigma_r^2} [\frac{1}{2} ||\mathbf{o}_r - |\mathbf{u}_r|^2||^2 +$$

$$\frac{1}{\gamma_r} ||\mathbf{u}_r - \mathbf{A}_r \cdot \mathbf{u}_0||^2 + \quad (7)$$

$$\frac{2}{\gamma_r} \operatorname{Re}\{\mathbf{\Lambda}_r^{*T} (\mathbf{u}_r - \mathbf{A}_r \cdot \mathbf{u}_0)\}] + \mu ||\mathbf{u}_0||_2^2.$$

The Lagrangian based optimization is associated with the saddle problem, which requires minimization on $\mathbf{u}_0, \{\mathbf{u}_r\}$ and maximization on the vector of the Lagrange multipliers $\{\mathbf{\Lambda}_r\}$. The parameters γ_r are positive.

In Augmented Lagrangian both the linear and quadratic terms in (7) correspond to the linear constrains $\mathbf{u}_r - \mathbf{A}_r \cdot \mathbf{u}_0 = 0$. If we keep only the quadratic terms the augmented Lagrangian becomes the penalty criterion, which assumes that the penalty coefficients $1/\gamma_r$ are large. As a rule it leads to computational difficulties because this criterion can be very ill-conditioned. If we keep only the linear terms the augmented Lagrangian becomes the standard Lagrangian. However, the saddle-point of this standard Lagrangian is unstable. It results in the problems with numerical solutions. The stability of the saddle-point of the augmented Lagrangian keeping both the linear and quadratic terms is one of the principal advantages of this criterion.

The proposed algorithm corresponds to a recursive solution of the problem $\max_{\{\mathbf{\Lambda}_r\}} \min_{\mathbf{u}_0, \{\mathbf{u}_r\}} L(\mathbf{u}_0, \{\mathbf{u}_r\}, \{\mathbf{\Lambda}_r\})$.

It is composed from the following successive steps :

AL Algorithm (8)

1. Set $t = 0$ (initialization), $\mathbf{u}_{0,0}$, $\mathbf{\Lambda}_{r,0}$,
2. Repeat, $t = 0, 1, ...,$
3. $\mathbf{u}_{r,t+1/2} = \mathbf{A}_r \cdot \mathbf{u}_{0,t}$,
4. $\mathbf{u}_{r,t+1}[k] = \mathcal{G}(\mathbf{o}_r[k], \mathbf{u}_{r,t+1/2}[k], \mathbf{\Lambda}_{r,t}[k])$,
5. $\mathbf{u}_{0,t+1} = (\sum_{r=1}^{l} \frac{1}{\gamma_r \sigma_r^2} \mathbf{A}_r^{*T} \mathbf{A}_r + \mu \cdot \mathbf{I}_{n \times n})^{-1} \times$ $\sum_{r=1}^{l} \frac{1}{\gamma_r \sigma_r^2} \mathbf{A}_r^{*T} (\mathbf{u}_{r,t+1} + \mathbf{\Lambda}_{r,t})$,
6. $\mathbf{\Lambda}_{r,t+1} = \mathbf{\Lambda}_{r,t} + \frac{1}{\gamma_r} \cdot (\mathbf{u}_{r,t+1} - \mathbf{u}_{r,t+1/2})$,
7. Stop

In Step 4 $\mathcal{G}(\mathbf{o}_r[k], \mathbf{u}_{r,t+1/2}[k], \mathbf{\Lambda}_{r,t}[k])$ is a solution of $\min_{\mathbf{u}_r[k]} L(\mathbf{u}_{0,t}, \{\mathbf{u}_r\}, \{\mathbf{\Lambda}_{r,t}\})$. Step 5 updates the reconstruction of the object distribution $\mathbf{u}_{0,t+1}$, used in Step 3 for prediction of the wave field distribution in the rth plane. Step 6 defines the updates of the Lagrange multipliers.

IV. Simulation Experiments

We tested our algorithm in multiple experiments for amplitude, phase and complex-valued object wave fields. The experiments are produced for various distances between the object and sensor planes as well as between the sensor planes.

The algorithm demonstrates a good convergence rate, accuracy and visualization for the phase and amplitude reconstructions. The algorithm performance for both noiseless and noisy data is tested. The perfect reconstruction of the wave field can be achieved for noiseless data even if the distances between the object and sensor planes are much larger that the in-focus ones.

The comparison is produced versus the single-beam multiple-intensity phase reconstruction (SBMIR) algorithm developed in [3], [4]. The AL algorithm enables a better accuracy and good imaging sometimes even when the alternative technique fails.

In our simulation experiments the observations $\mathbf{o}_r$ are always generated by F-DDT. This modeling is accurate for the pixelated sensors and object distribution approximation [6]. It is an appropriate choice for simulation, where we deal only with pixelated object and sensor distributions.

For reconstruction we tested different models of $\mathbf{A}_r$ including the discrete convolution and angular spectrum decompositions of a single and double size image support. Overall, the best results are achieved using F-DDT for $\mathbf{A}_r$ in the AL algorithm.

More details concerning the proposed algorithm, implementation and experiments can be found in [11].

V. Conclusion

This section concerns our further research.

A. A priory on phase and module of the object distribution.

Often wave field distributions (imaging) allow sparse representations in transformed domains. In our days this sparsity is characterized by l_0-norm in the spectrum domain, i.e. as a number of active non-zero elements in the spectrum domain. If we use the notation $\mathbf{\Omega} \in \mathbb{R}^M$ for the corresponding vector-spectrum and $\mathbf{T}$ for the spectrum matrix transform, $\mathbf{\Omega} = \mathbf{T} \cdot \mathbf{u}_0$ the object distribution reconstruction problem (5) is replaced by a more complex one

$$\hat{\mathbf{\Omega}} = \arg\min_{\mathbf{\Omega}} J(\mathbf{u}_0), \quad (9)$$

$$\text{subject to } \mathbf{u}_{z_r} = \mathbf{A}_r \cdot \mathbf{u}_0,\ \mathbf{u}_0 = \mathbf{T} \cdot \mathbf{\Omega}, \quad (10)$$

$$J(\mathbf{u}_0) = \sum_{r=1}^{l} \frac{1}{2\sigma_r^2} ||\mathbf{o}_r - |\mathbf{u}_r|^2||^2 + \mu \cdot pen(\mathbf{\Omega}), \quad (11)$$

$$pen(\mathbf{\Omega}) = ||\mathbf{\Omega}||_0, \quad (12)$$

where the optimization is produced in spectrum domain over $\mathbf{\Omega}$.

This setting includes an automatics selection of adaptive models for $\mathbf{u}_0$, in particular efficient if the overcomplete transform is used with M larger (much larger) than the number of elements of $\mathbf{u}_0$.

The spectrum and the penalty can specified to be different for phase and module object distributions. In order to solve (9) we use a specially developed splitting augmented Lagrangian (SAL) algorithm where extra variables are introduced for splitting the spectrum and signal domain variables. For examples of this type of algorithms we refer to the work on *split Bregman iterations [12].*

A proper selection of the spectrum domain and the adaptivity of the algorithm enable potentially much better performance as it can be achieved using the AL algorithm, where the priory information on u_0 can be given only in the signal domain.

B. Backward propagation by variational inverse

The variational problems (5) and (9) are based on the forward propagation modeling only and do not use conventional backward propagation. The back propagation or inverse imaging is found as a solution of these variational problems enabling an optimal correspondence between the observations and the prediction for the observation planes obtained from the solution found for $\mathbf{u}_0$. The priory on the object distribution is automatically included in this variational backward propagation.

The variational inverse imaging can be used with any kind of forward propagation models, in particular, with the models found experimentally.

C. Generalizations and further developments

The considered variational setting can be generalized for various optical settings, for design of the object distribution giving desirable wave field distributions, for advance imaging techniques such as super-resolution and compressive sensing.

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